Is entanglement entropy proportional to area?

Morteza Ahmadi, Saurya Das, and S. Shankaranarayanan

Abstract: It is known that the entanglement entropy of a scalar field, found by tracing over its degrees of freedom inside a sphere of radius $\mathcal{R}$, is proportional to the area of the sphere (and not its volume). This suggests that the origin of black hole entropy, also proportional to its horizon area, may lie in the entanglement between the degrees of freedom inside and outside the horizon. We examine this proposal carefully by including excited states, to check probable deviations from the area law.

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Résumé : Nous savons que l'entropie d'entrelacement d'un champ scalaire, trouvée en suivant ses degrés de liberté à l'intérieur d'une sphère de rayon $\mathcal{R}$, est proportionelle à la surface de la sphère (et non à son volume). Ceci suggère que l'origine de l'entropie d'un trou noir, également proportionelle à la surface de son horizon, peut se trouver dans l'entrelacement entre les degrés de liberté à l'intérior et à l'extérior de l'horizon. Nous examinons avec soin cette hypothèse, en incluant les états excités, afin d'identifier les déviations possibles de la loi de surface.

[Traduit par la rédaction]

1. Introduction

There are strong indications that a black hole (BH) of mass M and horizon area A_H possesses entropy and temperature, given respectively by [1]:

$$S_{BH} = \frac{A_H}{4\ell_{Pl}^2}\,, \qquad T_H = \frac{\hbar c^3}{8\pi G M}\,, \qquad\qquad (\ell_{Pl} = \text{Planck length})\,. \tag{1}$$

The above entropy and temperature satisfy the laws of *BH Thermodynamics*

$$T_H = \text{Constant on horizon}\,,\ \ d\left(Mc^2\right) = T_H dS_{BH} + \text{work terms}\,,\ \ \Delta(S_{BH} + S_{matter}) \geq 0 \tag{2}$$

where 'work terms' are relevant for BHs with charge(s) and angular momenta, and S_{matter} refers to the entropy of matter outside the BH. Usually the latter can be given a microscopic interpretation by the relation $S_{matter} = \ln\Omega$, where Ω is the number of accessible micro-states, compatible with the various macroscopic parameters such as temperature, pressure and volume. Although such an interpretation for S_{BH} is incomplete, important progress has been made in various approaches to quantum gravity [2–5].

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This, coupled with the area proportionality of BH entropy (as opposed to volume proportionality), has raised a fundamental question in quantum gravity, namely: *What is the origin of BH entropy?* Potential candidates include strings, D-branes, spin-network states and conformal degrees of freedom at the horizon. Another popular (and also incomplete) approach to the BH entropy is entanglement entropy, which is a measure of the information loss due to the spatial separation between the degrees of freedom inside and outside the horizon. The so-called brick wall model has been a concrete realization of this idea, in which entanglement entropy of the scalar (and other) fields have been computed by tracing over their degrees of freedom outside the horizon [6][1]. The brick-wall entropy turns out to be proportional to the BH horizon area. The problem, however, is that due to the infinite growth of the density of states close to the horizon, one has to impose ultraviolet cut-off near the horizon and hence the entropy depends on the cut-off scale. This clearly is an undesirable feature.

A simpler physical system was considered in refs. [7, 8], in which the entropy of a scalar field on a suitably discretized *flat* space was computed numerically, by tracing over the degrees of freedom of a hypothetical sphere of radius $\mathcal{R}$. This gave the remarkable result that the entanglement entropy was indeed proportional to the area of the spherical surface (also see [9] for analytical proofs of the result). This further supported the idea that entanglement was responsible for BH entropy. One of the key assumptions in refs. [7, 8] was that the harmonic oscillators (HOs) resulting from the scalar field, on discretization of space, were *all* in their ground states. In this article, we would like to relax this assumption, and investigate the robustness of the entropy-area relation in these physical systems. As a step towards understanding of the entanglement entropy of N coupled harmonic oscillators (by tracing over $n < N$ degrees of freedom), in this work we consider two HOs ($N = 2, n = 1$) in two physically interesting limits: coherent states and superposition of excited and ground states. We show, numerically, that the presence of excited states results in an increase of entropy. The generalization to arbitrary N will be left to a future publication [10].

In the next section, we will briefly review the entanglement entropy for ground state HO. In section (3), we will generalize the results for the two HO wave-function which is a superposition of ground state and the first excited state, and show that the entropy increases. In the concluding section, we will remark on the possible implications of our results to the physically interesting case of a large number of oscillators, whose couplings are determined by the Lagrangian of a free scalar field.

We will follow the notations of ref. [8] to provide easy comparison, and henceforth use $\hbar = 1$.

2. Ground State Entanglement Entropy

The Hamiltonian for two coupled HOs of unit masses:

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + k_0\left(x_1^2 + x_2^2\right) + k_1\left(x_1 - x_2\right)^2\right] \tag{3}$$

has the ground state solution in terms of normal modes ($x_\pm = \frac{x_1 \pm x_2}{\sqrt{2}}$, $\omega_+ = \sqrt{k_0}, \omega_- = \sqrt{k_0 + 2k_1}$):

$$\psi_0\left(x_1, x_2\right) = \psi_0(x_+)\psi_0(x_-) = \frac{\left(\omega_+\omega_-\right)^{1/4}}{\pi^{1/2}} \exp\left[-\left(\omega_+ x_+^2 + \omega_- x_-^2\right)/2\right] . \tag{4}$$

When traced over the oscillator characterized by x_1, the resultant density matrix is:

$$\rho_0\left(x_2, x_2'\right) = \int_{-\infty}^{\infty} dx_1 \psi_0\left(x_1, x_2\right)\psi_0^\star\left(x_1, x_2'\right) = \sqrt{\frac{\gamma - \beta}{\pi}} \exp\left[-\gamma\left(x_2^2 + x_2'^2\right)/2 + \beta x_2 x_2'\right] , \tag{5}$$

[1] Note that tracing over the outside does not pose any conceptual problem, since for a pure system, tracing over a given subsystem and its complementary subsystem yield identical entropies [11].

whose eigenfunctions and eigenvalues, respectively are

$$f_n(x) = H_n\left(\sqrt{\alpha}x\right)\exp\left(-\alpha x^2/2\right), \qquad p_n = (1-\xi)\,\xi^n \ . \tag{6}$$

The above density matrix gives rise to the following entropy [7, 8]:

$$S(\xi) = -Tr\left(\rho_0 \ln \rho_0\right) = -\sum_{n=0}^{\infty} p_n \ln p_n = -\ln(1-\xi) - \frac{\xi}{1-\xi}\ln\xi\,, \tag{7}$$

where

$$R^2 \equiv \frac{\omega_+}{\omega_-} < 1, \quad \alpha = \omega_- R, \quad \beta = \frac{\omega_-(1-R^2)^2}{4(1+R^2)}, \quad \gamma = \frac{1+6R^2+R^4}{4(1+R^2)}, \quad \xi = \left(\frac{1-R}{1+R}\right)^2 . \tag{8}$$

Note that $R = 0, 1$ correspond to the strongly coupled and uncoupled limits respectively.

The Hamiltonian for a free, massless scalar field φ in flat space-time is given by:

$$H = \frac{1}{2}\int d^3x\left[\pi^2(\vec{r}) + |\nabla\varphi(\vec{r})|^2\right] . \tag{9}$$

Discretizing the space (a being the lattice spacing, Na signifying the infrared cutoff, and l, m are the parameters in the spherical harmonics $Y_{l,m}(\theta,\phi)$):

$$H = \sum_{lm} H_{lm} = \frac{1}{2a}\sum_{j=1}^{N}\left[\pi_{lm,j}^2 + \left(j+\frac{1}{2}\right)^2\left(\frac{\varphi_{lm,j}}{j} - \frac{\varphi_{lm,j+1}}{j+1}\right)^2 + \frac{l(l+1)}{j^2}\varphi_{lm,j}^2\right] . \tag{10}$$

The above expression for H_{lm} is a special case of the general N-coupled oscillator Hamiltonian:

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j \ . \tag{11}$$

The corresponding N-HO ground state wave function is given by:

$$\psi_0(x_1,\ldots,x_n,x_{n+1},\ldots,x_N) = \left[\frac{|\Omega|}{\pi^N}\right]^{1/4}\exp\left[-\frac{x^T\cdot\Omega\cdot x}{2}\right], \tag{12}$$

where $\Omega^2 = K$. It can be shown that for the ground state wave function, the density matrix (when one traces over $n < N$ oscillators) can be factorized into a product of $(N-n)$ 2-HO density matrices. Thus the total entropy is simply the sum of the entropies. For the scalar field, n is taken to be proportional to the radius of the sphere which is traced over, i. e., $\mathcal{R} = (n+1/2)a$. For the Hamiltonian (10) the interaction matrix K_{ij} can be read-off, resulting in the entanglement entropy

$$S = 0.3(n+1/2)^2 \propto \mathcal{R}^2 \ , \tag{13}$$

signifying area proportionality. It is worth noting, again, that all the oscillators are assumed to be in their ground states.

3. Entanglement entropy of excited states:

Let us now consider the excited states of the N HOs discussed in the previous section. The corresponding wave-function is:

$$\psi(x_1,\ldots,x_n,x_{n+1},\ldots,x_N) = \left[\frac{|\Omega|}{\pi^N}\right]^{1/4}\exp\left[-\frac{x^T\cdot\Omega\cdot x}{2}\right]\prod_{i=1}^{N}\frac{1}{\sqrt{2^{\nu_i}\nu_i!}}\,H_{\nu_i}\left(K_{Di}^{\frac{1}{4}}\,\underline{\mathrm{x}}_i\right) , \tag{14}$$

where $K_D \equiv UKU^T$ is a diagonal matrix ($U^TU = I_N$), $\underline{\mathrm{x}} \equiv Ux$ and ν_i $(i = 1 \ldots N)$ are indices of the Hermite polynomials. The density matrix, tracing over first n of N oscillators, is

$$\rho_0\left(x_{n+1}, \ldots, x_N; x'_{n+1}, \ldots, x'_N\right) = \left[\frac{|\Omega|}{\pi^N}\right]^{1/2} \int \prod_{i=1}^{n} dx_i \exp\left[-\frac{x^T \cdot \Omega \cdot x}{2}\right] \times$$
$$\prod_{i=1}^{N} \frac{1}{\sqrt{2^{\nu_i}\nu_i!}}\, H_{\nu_i}\left(K_{D_i}^{\frac{1}{4}}\, \underline{\mathrm{x}}_i\right) \exp\left[-\frac{x'^T \cdot \Omega \cdot x'}{2}\right] \prod_{j=1}^{N} \frac{1}{\sqrt{2^{\nu_j}\nu_j!}}\, H_{\nu_j}\left(K_{D_j}{}^{\frac{1}{4}}\, \underline{\mathrm{x}}'_j\right). \tag{15}$$

The evaluation of the integral of the product of $2N$ Hermite polynomials, although may not be impossible, is in general, non-trivial. In order to keep the calculations simple, we consider two specific physical cases: (i) coherent states and (ii) superposition of ground and first excited states.

The coherent states, which are eigenstates of the harmonic oscillator annihilation operator with real eigenvalues, are described by the following wave function:

$$\psi_{CS}(x, a) \equiv \psi_0(x - a) = e^{-i\hat{p}a}\, \psi_0(x)\,. \tag{16}$$

The expectation of the position operator, w.r.t the coherent state wave function, oscillates in time with an amplitude a and the state has the minimum allowable uncertainty

$$\Delta p\, \Delta x = \frac{1}{2}\,, \tag{17}$$

same as that of the ground state. For two coupled oscillators, the corresponding coherent state is:

$$\psi_{CS}(x_1, x_2) \equiv \psi_{CS}(x_+, a)\psi_{CS}(x_-, b) = \psi_0(x_+ - a)\psi_0(x_- - b)\,. \tag{18}$$

Defining $\tilde{x}_2 = x_2 - (a - b)/\sqrt{2}$, it is easy to show that the corresponding density matrix retains the same form as (5), albeit in terms of these new variables:

$$\rho_{CS}(x_2, x'_2) = \rho_{out}\left(\tilde{x}_2, \tilde{x}'_2\right)\,. \tag{19}$$

Thus, from Eqs. (6), it follows that the eigenfunctions are $f_n(\tilde{x})$ and eigenvalues remain unchanged (p_n), and we get the interesting result that the entropy is the same as that for the ground state! Presumably, this is because of the fact that coherent states are obtained by translating the ground state in phase space. The result can be easily generalized to an arbitrary number of harmonic oscillators [10].

Next, we consider the superposition of the ground and first excited state of the 2-HO system:

$$\psi(x_1, x_2) = \alpha_1\, \psi_1(x_+)\psi_0(x_-) + \beta_1\, \psi_0(x_+)\psi_1(x_-) + \gamma_1\, \psi_0(x_+)\psi_0(x_-) \quad [\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1], \tag{20}$$

$$\text{where} \quad \psi_{\rm n}({\rm x}) = {\rm N_n}(\omega)\, {\rm e}^{-\omega^2 {\rm x}^2/2} {\rm H_n}(\sqrt{\omega}\, {\rm x}),\ {\rm N_n}(\omega) = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^{\rm n}{\rm n}!}} \tag{21}$$

is the n^{th} excited state of an oscillator. Although from the identity of particles one would expect $\alpha_1 = \beta_1$, we do not impose such a condition at this point. From (15), the density matrix follows:

$$\rho(x_2, x'_2) = \rho_0(x_2, x'_2)[A\,(x_2^2 + x_2'^2) + B\, x_2 x'_2 + C\,(x_2 + x'_2) + D]\,, \tag{22}$$

where $\rho_0(x_2, x'_2)$ is the ground state density matrix given by Eq. (5), and the constants are given as:

$$\begin{aligned} &A = \alpha_1^2 a + \beta_1^2 a_3 + \alpha_1\beta_1 a_4\,, \qquad B = \alpha_1^2 b + \beta_1^2 b_3 + \alpha_1\beta_1 b_4\,, \qquad C = \gamma_1(\alpha_1 a_6 + \beta_1 a_7)\,, \\ &D = \alpha_1^2 c + \beta_1^2 c_3 + \alpha_1\beta_1 c_4 + \gamma^2\,, \qquad a_6 = \frac{2\sqrt{\omega_-}\, R}{1 + R^2}\,, \qquad a_7 = -\frac{2\sqrt{\omega_-}\, R^2}{1 + R^2}\,, \\ &a = \frac{R^2(1 - R^2)(3 + R^2)\,\omega_-}{4(1 + R^2)^2}\,, \qquad b = \frac{R^2(5 + 2R^2 + R^4)\,\omega_-}{2(1 + R^2)^2}\,, \qquad c = \frac{R^2}{1 + R^2}\,, \end{aligned} \tag{23}$$

$$a_3 = -\frac{(1-R^2)(1+3R^2)\omega_-}{4(1+R^2)^2}\,, \qquad b_3 = \frac{(1+2R^2+5R^4)\omega_-}{2(1+R^2)^2}\,, \qquad c_3 = \frac{1}{1+R^2}\,,$$
$$a_4 = \left(\frac{1-R^2}{1+R^2}\right)^2 \frac{\omega_- R}{2}\,, \qquad b_4 = -\frac{R(1+6R^2+R^4)\omega_-}{(1+R^2)^2}\,, \qquad c_4 = \frac{2R}{1+R^2}\,.$$

It can be verified that: $Tr(\rho) = \int_{-\infty}^{\infty} dx_2\, \rho(x_2, x_2) = \alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1$. To find the eigenvalues of the density matrix (22), we follow the general procedure outlined in ref. [12]. First, we expand $\rho(x_2, x_2')$ in terms of general HO eigenstates (although, in principle any complete set of functions should suffice):

$$\rho(x_2, x_2') = \sum_{n=0}^{\infty} h_n(x_2) g_n(x_2')\,,\; h_m(x_2) = N_m(\alpha) \exp\left(-\frac{\alpha x^2}{2}\right) H_m(\sqrt{\alpha}\, x_2)\,. \tag{24}$$

Inverting, we get:

$$\begin{aligned} g_m(x_2') &= \int_{-\infty}^{\infty} dx_2\, \rho(x_2, x_2')\, h_m(x_2) \qquad (25)\\ &= p_m N_m e^{-\frac{\gamma x_2'^2}{2}\frac{(\beta x_2')^2}{2(\gamma+\alpha)}} \left[\left(B_1 x_2' + E_1\right) H_{m+1}(\sqrt{\alpha}\, x_2') + \left(C_1 x_2'^2 + D_1 + F_1 x_2'\right) H_m(\sqrt{\alpha}\, x_2')\right]\,, \end{aligned}$$

where

$$B_1 = -\sqrt{\alpha}\left[\frac{2\bar{a}\gamma}{\gamma^2-\alpha^2} + \frac{\bar{b}}{\sqrt{\gamma^2-\alpha^2}}\right], \qquad C_1 = \frac{2\bar{a}\gamma}{\gamma-\alpha} + \bar{b}\sqrt{\frac{\gamma+\alpha}{\gamma-a}}\,, \qquad D_1 \equiv D_{11} + D_{12},$$
$$D_{11} = \frac{\bar{a}}{\gamma+\alpha} + \bar{c}\,, \qquad D_{12} = -\frac{2\bar{a}\alpha}{\gamma^2-\alpha^2}\,, \qquad E = -\frac{\bar{d}\sqrt{\alpha}}{\sqrt{\gamma^2-\alpha^2}}\,, \qquad F_1 = \bar{d}\left[1 + \sqrt{\frac{\gamma+\alpha}{\gamma-\alpha}}\right] \tag{26}$$
$$\bar{a} = \alpha_1^2 a + \beta_1^2 a_3 + \alpha_1\beta_1 a_4,\; \bar{b} = \alpha_1^2 b + \beta_1^2 b_3 + \alpha_1\beta_1 b_4,\; \bar{c} = \alpha_1^2 c + \beta_1^2 c_3 + \alpha_1\beta_1 c_4 + \gamma^2,\; \bar{d} = \alpha_1 a_6 + \beta_1 a_7\,,$$

and a_i, b_i have been defined in Eq. (23). The next step is to define the matrix equivalent of ρ, i. e.,

$$\begin{aligned} \alpha_{pm} &\equiv \int_{-\infty}^{\infty} dx\, g_m(x) h_p(x) \qquad (27)\\ &= p_m \left[\left(D_{11} + pD_{12} + \frac{B_1(p+1)}{\sqrt{\alpha}} + \frac{C_1(2p+1)}{2\alpha}\right)\delta_{pm} + \frac{C_1}{2\alpha}\sqrt{(p+1)(p+2)}\delta_{p,m-2}\right.\\ &+ \left.\sqrt{p(p-1)}\left(\frac{B_1}{\sqrt{\alpha}} + \frac{C_1}{2\alpha}\right)\delta_{p,m+2} + F_1\sqrt{\frac{p+1}{2\alpha}}\delta_{p+1,m} + \left(E_1\sqrt{2p} + F_1\sqrt{\frac{p}{2\alpha}}\delta_{p-1,m}\right)\delta_{p-1,m}\right]. \end{aligned}$$

Although formally diagonalizable, the eigenvalues λ_p of the above penta-diagonal matrix are most easily found numerically. With MAPLE, using upto 40×40 matrices, we verified that it has unit trace.

$$Tr(\alpha_{pm}) = \sum_{m=0}^{\infty} \alpha_{mm} = 1\,, \qquad \alpha_{mm} = p_m\left[\left(D_1 + \frac{B_1}{\sqrt{\alpha}} + \frac{C_1}{2\alpha}\right) + m\left(D_{12} + \frac{B_1}{\sqrt{\alpha}} + \frac{C_1}{\alpha}\right)\right] .\tag{28}$$

The corresponding entropy as function of α_1, β_1, R defined as:

$$S(\alpha_1, \beta_1, R) = -\sum_{p=0}^{\infty} \lambda_p \ln \lambda_p \tag{29}$$

was also computed numerically, and for all $\alpha_1, \beta_1 \neq 0$ it was found that $S(\alpha_1, \beta_1, R) \geq S(0, 0, R)$, where $S(0, 0, R)$ is the ground state entropy. The equality holds *only* in the uncoupled limit $R = 1$ and $\alpha_1 = \beta_1$. These features are visible in Fig.(1), where we have plotted entropies for the excited state

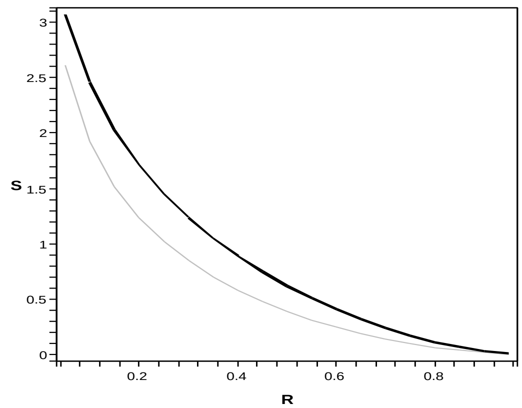

Fig. 1. Plots of the entanglement entropy of the excited state $S(1/\sqrt{2}, 1/\sqrt{2}, R)$ (black curve) and that of the ground state $S(0, 0, R)$ (grey curve) vs. R. Note that $S(1/\sqrt{2}, 1/\sqrt{2}, R) > S(0, 0, R)$ for all $R < 1$.

$[\alpha_1 = \beta_1 = 1/\sqrt{2}, \gamma_1 = 0]$ as well as the ground state. In brief, any amount of excited state in the superposition increases the entropy. This is intuitively expected, since it can be shown that the expectation of energy in the state (20) is give by: $< H(\alpha_1, \beta_1, \gamma_1) >= \frac{\omega_-}{2} \left[\alpha_1^2(1 + 3R^2) + \beta_1^2(3 + R^2) + \gamma_1^2(1 + R^2)\right]$, from which it follows that $< H(\alpha_1, \beta_1, \gamma_1) > - < H(0, 0, 1) >= \frac{\omega_-}{2} \left(\alpha_1 R^2 + \beta_1^2\right) \geq 0$. That is, the expectation of energy is least for the ground state; and higher energies are normally associated with higher entropies.

4. Summary and Outlook

In this paper, we have shown that the entanglement entropy of two coupled HOs, with coordinates of one traced over, is more for excited states compared to when they are both in their ground states. We would like to extend our results to N oscillators, with $n < N$ of them being traced over. This would enable one to compute the entanglement entropy of a free scalar field in flat space-time when its degrees of freedom inside a given region are traced over, and check whether it is proportional to the area of the bounding surface [10]. It would also be interesting to extend the results to BH space-times with the surface mentioned above coinciding with its event horizon. We hope to report on these and related issues in the near future.

Note added: After the submission of this paper to the *Canadian Journal of Physics*, we extended the work (reported here) to the free scalar field in flat space-time in Ref. [10]. We have shown that the entanglement entropy is proportional to area when the scalar field degrees of freedom are in generic coherent states, and first excited state, although in the latter case, the entropy increases manyfold.

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